SMOOTH MANIFOLDS FALL 2023 - HOMEWORK 2

Problem 1. Consider the transformation group $\Gamma = \left\{R_{k\pi/2} : k \in \mathbb{Z}\right\}$, where $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation determined by the matrix $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Show that the quotient space $\mathbb{R}^2/\Gamma$ is a topological manifold. Is it homeomorphic to a familiar one? Furthermore, does it have a smooth structure such that the map $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\Gamma$ is a submersion? Does there exist a smooth structure such that π is C^∞? Justify your answers through pictures and "moral" arguments: do not write formal proofs.

Problem 2. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus, and assume that $D \subset \mathbb{T}^2$ is a discrete subgroup. Show that $\mathbb{T}^2/D$ is diffeomorphic to $\mathbb{T}^2$. [*Hint*: Show that the lift of D to $\mathbb{R}^2$ is also a discrete subgroup containing $\mathbb{Z}^2$. You may use the fact that all discrete subgroups of $\mathbb{R}^2$ are isomorphic to $\{0\}$ $\mathbb{Z}$, and $\mathbb{Z}^2$.]

Problem 3. Let $\varphi : U \to \mathbb{R}^n$ be a smooth chart for a smooth manifold M. Define a corresponding set $\hat{U} = \bigcup_{p\in U} T_pM \subset TM$, and let an element of $\hat{U}$ be denoted by v_p, where p denotes the basepoint of the vector v. Define $\hat{\varphi} : \hat{U} \to \mathbb{R}^n \times \mathbb{R}^n$ by

$$\hat{\varphi}(v_p) = (\varphi(p), v_{p,\varphi}),$$

where $v_{p,\varphi} \in T_p^\varphi M := T_{\varphi(p)}\mathbb{R}^n = \mathbb{R}^n$ is the vector v as represented in the chart φ (recall that T_pM is isomorphic to each $T_p^\varphi M$ for any chart φ whose domain contains p). Show that if $\mathcal{A}$ is a smooth atlas of charts for M, then $\hat{\mathcal{A}} = \left\{(\hat{U}, \hat{\varphi}) : (U, \varphi) \in \mathcal{A}\right\}$ is a smooth atlas on TM (this is the *smooth atlas on TM induced by $\mathcal{A}$*).

Problem 4. Show that if $U \subset \mathbb{R}^k$ is open, $\varphi : U \to \mathbb{R}^n$ is C^∞ and $d\varphi(x)$ is injective, then there exists a C^∞ change of coordinates diffeomorphism $H : \mathbb{R}^n \to \mathbb{R}^n$ defined near x such that $H \circ \varphi$ takes values in $\mathbb{R}^k \subset \mathbb{R}^n$.

Problem 5. Let M be a C^∞ manifold, $k \in \mathbb{N}$ and $X \subset M$ be a closed, connected subset such that for every $x \in X$, there exists $U \subset \mathbb{R}^k$ and an embedding $\varphi : U \to M$ such that the image of φ is an open neighborhood of x in X. Show that X has a C^∞ manifold structure such that the inclusion of X into M is an embedding. [*Hint*: Use the previous problem!]